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THE RECOVERY OF BOUNDARY DATA FOR THE EDDY-CURRENT PROBLEM ON POLYHEDRA : NUMERICAL APPROACH

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Abstract - We consider a boundary data identification problem in quasi-static electromagnetism. Two kinds of boundary conditions are simultaneously imposed on one part of the boundary. The other part of the boundary is unreachable, i.e. the boundary condition is unknown and it has to be determined as a part of the problem. We assume that the permeability outside the domain satisfies $\mu \gg \mu_0$. Two different numerical approaches to solve this problem are presented, compared and their results are given.

1. INTRODUCTION

In many problems ranging from physics, mechanics, geology to medical sciences, one has to solve partial differential equations, where some unknown coefficients are involved. In these problems one tries to recover these unknown coefficients via boundary measurements of certain quantities. There are different ways how to do it, e.g., minimization methods, alternating methods etc.

We deal with an identification problem of an unknown boundary data and the determination of a solution inside of a domain from the measurements on a part of the boundary. The lack of information on the unreachable part of the boundary has to be compensated by over-determined data on the reachable part. We consider this problem particularly for a case of a quasi-static model in electromagnetism. This model, also known as eddy-current model for Maxwell's equations, neglects the displacement currents. It is feasible for low-frequency simulations as design of transformers. The justification of this model can be found, e.g., in [1]. The numerical solution of this model asks much less computational effort than that of complete Maxwell's system.

We design two different approaches to solve the problem described above. The first one is the so-called "method of answers" (MOA). On the unreachable part of the boundary we invoke different impulses and we compute their "answers" on the reachable part. Due to the linearity of the problem, we search for a linear combination of "answers" satisfying the over-determined data on the reachable part. This leads to a linear system of algebraic equations for coefficients. Such a system is generally ill-conditioned.

The second method called the "Adjoint Method" (AM) is based on an iterative approach, particularly the steepest descent method. We construct a regularized functional and its gradient. We use the variational framework. We aim to work in natural Sobolev spaces on a polyhedra keeping the regularity of a solution as low as possible.

Both approaches, mentioned above, require solution of direct problems. The discretization is done by means of edge finite elements, which are natural for magnetic fields (see [2]).

2. NOTATIONS AND PROBLEM SETTING

Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz polyhedra with the boundary Γ and let $\boldsymbol{n}$ be the outward normal to Γ. We consider a linear, isotropic, homogeneous medium. In this case Maxwell's equations can be written in the following form :

$$\left.\begin{array}{rcrcl} \mu\partial_t \boldsymbol{H} & + & \nabla\wedge\boldsymbol{E} & = & \boldsymbol{0} \\ -\varepsilon\partial_t\boldsymbol{E} & + & \nabla\wedge\boldsymbol{H} & = & \sigma\boldsymbol{E} \\ & & \nabla\cdot\boldsymbol{E} & = & \dfrac{\rho}{\varepsilon} \\ & & \nabla\cdot\boldsymbol{H} & = & 0 \end{array}\right\} \quad \text{in } \Omega\times(0,T), \tag{1}$$

where $\boldsymbol{H}$ and $\boldsymbol{E}$ denote the magnetic and electric fields, μ, σ, ε, ρ are the permeability, conductivity, permittivity and electric charge density, respectively. In low-frequency applications one can assume that $\frac{\varepsilon}{\sigma}\partial_t\boldsymbol{E} \ll 1$. This condition is called *the quasi static condition for electric field.* The elimination of $\boldsymbol{E}$ in (1) leads to the following model for magnetic field :

$$\begin{array}{rcl} \partial_t\boldsymbol{H} + \mu^{-1}\sigma^{-1}\nabla\wedge\nabla\wedge\boldsymbol{H} & = & \boldsymbol{0} \\ \nabla\cdot\boldsymbol{H} & = & 0. \end{array} \tag{2}$$

From now on, we will call this system *the quasi static model for magnetic field.* When the time discretization is applied, the following system at each time point of a suitable time partitioning has to be solved :

$$\left.\begin{array}{rcl} \boldsymbol{H}+\alpha\nabla\wedge\nabla\wedge\boldsymbol{H} &=& \boldsymbol{f} \\ \nabla\cdot\boldsymbol{H} &=& 0 \end{array}\right\} \quad \text{in } \Omega, \qquad \text{for } \alpha\in\mathbb{R}. \tag{3}$$

To be accurate we have to solve :

$$\left.\begin{array}{rcl} \boldsymbol{H}_n+\Delta t\mu^{-1}\sigma^{-1}\nabla\wedge\nabla\wedge\boldsymbol{H}_n &=& \boldsymbol{H}_{n-1} \\ \nabla\cdot\boldsymbol{H}_n &=& 0 \end{array}\right\} \quad \text{in } \Omega, \tag{4}$$

where Δt is the time step and $\boldsymbol{H}_n$ is the solution in time $t=n\Delta t$.

If $\nabla\cdot\boldsymbol{f}=0$, which will be our case, we can omit $\nabla\cdot\boldsymbol{H}=0$ in (3). Note, that the assumption of the homogeneity says that μ,σ are constants. If we did not assume that, we would have the following problem instead of (2):

$$\begin{array}{rcl} \partial_t\boldsymbol{H}+\mu^{-1}\nabla\wedge(\sigma^{-1}\nabla\wedge\boldsymbol{H}) &=& \mathbf{0} \\ \nabla\cdot(\mu\boldsymbol{H}) &=& 0 \end{array} \tag{5}$$

Most of the results here can be generalized for (5). However, for simplicity we assume that μ,σ are given constants. Two natural types of boundary conditions are frequently used in direct problems :

$$\begin{array}{rcl} \boldsymbol{H}\wedge\boldsymbol{n} &=& \vec{H} \\ & \text{or} & \\ \nabla\wedge\boldsymbol{H}\wedge\boldsymbol{n} &=& \vec{C}. \end{array} \tag{6}$$

The first one prescribes the tangential component of a magnetic field and the second one imposes the information about the tangential component of an electric field.

We are going to work in a variational framework. We will use the following notation :

$$\begin{array}{rcl} \mathbf{L}^2(\Omega) &:=& \{\boldsymbol{u}\in \mathrm{L}^2(\Omega)^3\} \\ \mathbf{H}(\mathbf{curl},\Omega) &:=& \{\boldsymbol{u}\in\mathbf{L}^2(\Omega) : \mathbf{curl}\,\boldsymbol{u}\in\mathbf{L}^2(\Omega)\} \\ \mathbf{L}^2_t(\Gamma) &:=& \{\boldsymbol{u}\in\mathbf{L}^2(\Gamma) : \boldsymbol{u}\cdot\boldsymbol{n}|_\Gamma=0\}; \quad <\cdot,\cdot>_t \text{ its scalar product.} \end{array}$$

In $\mathbf{H}(\mathbf{curl},\Omega)$ we take the norm $\|\boldsymbol{H}\|_{\mathbf{H}(\mathbf{curl},\Omega)}=(\|\boldsymbol{H}\|^2_{\mathrm{L}^2(\Omega)}+\|\mathbf{curl}\,\boldsymbol{H}\|^2_{\mathrm{L}^2(\Omega)})^{1/2}$.

We will apply the theory developed in [3,4], where the authors introduced the traces of weak solutions of Maxwell's equations in Lipschitz polyhedra. A short summary of the necessary notations and results of these two articles follows.

Let us set:

$$\mathbf{H}^{1/2}_-(\Gamma):=\{\boldsymbol{u}\in\mathbf{L}^2_t(\Gamma):\boldsymbol{u}|_{\Gamma_j}\in\mathbf{H}^{1/2}(\Gamma_j), 1\le j\le N\},$$

where Γ_j are faces of a polyhedra.

Definition 1. *The "tangential components trace" mapping $\pi_\tau : \mathrm{C}^\infty(\overline{\Omega})\to\mathbf{H}^{1/2}_-(\Gamma)$ and the "tangential trace" mapping $\gamma_\tau : \mathrm{C}^\infty(\overline{\Omega})\to\mathbf{H}^{1/2}_-(\Gamma)$ are defined as $\boldsymbol{u}\mapsto\boldsymbol{n}\wedge(\boldsymbol{u}\wedge\boldsymbol{n})|_\Gamma$ and $\boldsymbol{u}\mapsto\boldsymbol{u}\wedge\boldsymbol{n}|_\Gamma$, respectively.*

These mappings can be extended by continuity to linear continuous mappings from $\mathbf{H}^1(\Omega)$ to $\mathbf{H}^{1/2}_-(\Gamma)$, and consequently they can be considered as mappings from $\mathbf{H}^{1/2}(\Gamma)$ to $\mathbf{H}^{1/2}_-(\Gamma)$.

Let Γ_+ be an nonempty open connected subset of Γ with a piecewise smooth boundary $\partial\Gamma$ and let $\Gamma_-=\Gamma\backslash\overline{\Gamma}_+$ be an nonempty complement of Γ_+. Let us set:

$$\begin{array}{lcl} \mathrm{H}^{1/2}_{00}(\Gamma_+) &:=& \{\varphi\in\mathrm{H}^{1/2}(\Gamma_+):\widetilde{\varphi}\in\mathrm{H}^{1/2}(\Gamma)\} \\ \mathrm{H}^{-1/2}_{00}(\Gamma_+) &:=& (\mathrm{H}^{1/2}_{00}(\Gamma_+))' \\ \mathbf{H}_{0,\Gamma_-}(\mathbf{curl},\Omega) &:=& \{\boldsymbol{u}\in\mathbf{H}(\mathbf{curl},\Omega):\boldsymbol{u}\wedge\boldsymbol{n}|_{\Gamma_-}=\mathbf{0} \text{ in } H^{-1/2}_{00}(\Gamma)^3\}, \end{array}$$

where $\widetilde{\varphi}$ is the prolongation by zero to the whole Γ. Now, we state **Theorem 6.6** from [4].

Theorem 1. *The mapping $\gamma^+_\tau : \mathbf{H}(\mathbf{curl},\Omega)\to\mathbf{H}^{-1/2}_{\|,00}(\mathrm{div}_{\Gamma_+},\Gamma_+)$ (respectively its restriction $\gamma^{+,0}_\tau$: $\mathbf{H}_{0,\Gamma_-}(\mathbf{curl},\Omega)\to\mathbf{H}^{-1/2}_{\|}(\mathrm{div}^0_{\Gamma_+},\Gamma_+)$) which associates to a vector field $\boldsymbol{u}\in\mathbf{H}(\mathbf{curl},\Omega)$ (resp. to $\boldsymbol{u}\in\mathbf{H}_{0,\Gamma_-}(\mathbf{curl},\Omega)$) its tangential components on Γ_+, that is $\boldsymbol{u}\wedge\boldsymbol{n}|_{\Gamma_+}$, is linear continuous and admits a continuous inverse.*

Analogical theorem holds true also for π_τ and of course also for the case of the whole boundary. The exhaustive definitions of the spaces $\mathbf{H}_{\|}^{-1/2}(\mathrm{div}^0_{\Gamma_+},\Gamma_+)$ and $\mathbf{H}_{\|,00}^{-1/2}(\mathrm{div}_{\Gamma_+},\Gamma_+)$, which are the suitable subspaces of $\mathbf{H}_{-}^{1/2}(\Gamma)$ taking into account special boundary conditions, exceed the scope of this article. They can be found in [3,4]. The reader should be aware of the fact, that according to **Theorem 1**, these subspaces of $\mathbf{H}_{-}^{1/2}(\Gamma)$ are such that the tangential trace mapping γ_τ^+ and its restriction $\gamma_\tau^{+,0}$ are linear, continuous and surjective. They are the equivalents of

$$\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma) := \{\boldsymbol{\lambda}\in H^{-1/2}(\Gamma)^3 : \boldsymbol{\lambda}\cdot\boldsymbol{n}=0, \mathrm{div}_\Gamma\boldsymbol{\lambda}\in H^{-1/2}(\Gamma)\} = \{\boldsymbol{u}\wedge\boldsymbol{n} : \boldsymbol{u}\in\mathbf{H}(\mathbf{curl},\Omega)\},$$

which is the trace space for $\mathbf{H}(\mathbf{curl},\Omega)$ when the domain Ω is *regular*. Here the differential operator div_Γ is defined by duality [6]:

$$\langle \mathrm{div}_\Gamma \boldsymbol{v}, \xi\rangle = \langle \boldsymbol{v}, \nabla_\Gamma\xi\rangle \quad \forall\, \xi \ \textit{regular},$$

where $\nabla_\Gamma\xi = \pi_\tau(\nabla\xi)$.

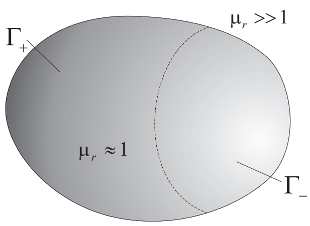


Figure 1: Domain Ω.

We assume that the relative permeability $\mu_r \approx 1 (=\mu/\mu_0)$ in Ω and $\mu_r \gg 1$ in $\mathbb{R}^3\backslash\bar{\Omega}$. In this case we can set $\boldsymbol{H}\approx\mathbf{0}$ in $\mathbb{R}^3\backslash\bar{\Omega}$. Such a simplification implies that we can consider a magnetic field to be living just inside the domain and we can neglect what is happening outside.
We would like to solve a special boundary identification problem for (3), namely: We prescribe simultaneously both types of the boundary conditions from (6) on a part of the boundary $\Gamma_+\subset\Gamma$ (determined). The problem is to identify the missing boundary data on $\Gamma_-\subset\Gamma$ (identify) together with magnetic field $\boldsymbol{H}$ in Ω satisfying the following problem.

Problem 1. *Find $\boldsymbol{H}\in\mathbf{H}(\mathbf{curl},\Omega)$ in Ω such that*

$$\begin{aligned} \boldsymbol{H} + \alpha\nabla\wedge\nabla\wedge\boldsymbol{H} &= \boldsymbol{f} && \textit{in } \Omega, \quad \textit{for } \alpha\in\mathbb{R}_+ \\ \left.\begin{aligned}\boldsymbol{H}\wedge\boldsymbol{n} &= \vec{H}\\ \nabla\wedge\boldsymbol{H}\wedge\boldsymbol{n} &= \vec{C}\end{aligned}\right\} & && \textit{on } \Gamma_+, \end{aligned} \tag{7}$$

where $\vec{H}$, $\vec{C}$ are given boundary data and the given $\boldsymbol{f}$ is divergence free.

We have to enforce some necessary conditions on $\vec{H}$ and $\vec{C}$. Suppose that (7) is uniquely solvable. Then it can be shown that not only $\boldsymbol{H}$ but also $\nabla\wedge\boldsymbol{H}$ belong to $\mathbf{H}(\mathbf{curl},\Omega)$. This together with **Theorem 1** implies that $\gamma_\tau^+(\boldsymbol{H})\in\mathbf{H}_{\|,00}^{-1/2}(\mathrm{div}_{\Gamma_+},\Gamma_+)$ and $\gamma_\tau^+(\nabla\wedge\boldsymbol{H})\in\mathbf{H}_{\|,00}^{-1/2}(\mathrm{div}_{\Gamma_+},\Gamma_+)$. Thus we have the necessary conditions that both $\vec{H}$ and $\vec{C}$ have to belong to $\mathbf{H}_{\|,00}^{-1/2}(\mathrm{div}_{\Gamma_+},\Gamma_+)$.

3. 1^{st} APPROACH - METHOD OF ANSWERS

As before, the fact that $\nabla\wedge\boldsymbol{H}\in\mathbf{H}(\mathbf{curl},\Omega)$ implies $\gamma_\tau^-(\nabla\wedge\boldsymbol{H})\in\mathbf{H}_{\|,00}^{-1/2}(\mathrm{div}_{\Gamma_-},\Gamma_-)$. According to the linear character of (7), we can apply the method of linear superposition. First, for any $\boldsymbol{\omega}\in\mathbf{H}_{\|,00}^{-1/2}(\mathrm{div}_{\Gamma_-},\Gamma_-)$ we define the following two auxiliary problems:

$$\begin{aligned} \boldsymbol{H}_1(\boldsymbol{\omega}) + \alpha\nabla\wedge\nabla\wedge\boldsymbol{H}_1(\boldsymbol{\omega}) &= \mathbf{0} && \text{in} && \Omega,\ \alpha\in\mathbb{R}_+ \\ \nabla\wedge\boldsymbol{H}_1(\boldsymbol{\omega})\wedge\boldsymbol{n} &= \mathbf{0} && \text{on} && \Gamma_+ \\ \nabla\wedge\boldsymbol{H}_1(\boldsymbol{\omega})\wedge\boldsymbol{n} &= \boldsymbol{\omega} && \text{on} && \Gamma_-, \end{aligned} \tag{8}$$

and

$$\begin{aligned} \boldsymbol{H}_2 + \alpha\nabla\wedge\nabla\wedge\boldsymbol{H}_2 &= \boldsymbol{f} && \text{in} && \Omega,\ \alpha\in\mathbb{R}_+ \\ \nabla\wedge\boldsymbol{H}_2\wedge\boldsymbol{n} &= \vec{C} && \text{on} && \Gamma_+ \\ \nabla\wedge\boldsymbol{H}_2\wedge\boldsymbol{n} &= \mathbf{0} && \text{on} && \Gamma_-. \end{aligned} \tag{9}$$

Let's define $\boldsymbol{H}(\boldsymbol{\omega}) := \boldsymbol{H}_1(\boldsymbol{\omega}) + \boldsymbol{H}_2$. Note, that $\boldsymbol{H}(\boldsymbol{\omega})$ satisfies the first and the third equation of (7). Now, take the mapping $T:\mathbf{H}_{\|,00}^{-1/2}(\mathrm{div}_{\Gamma_-},\Gamma_-)\to\mathbf{H}_{\|,00}^{-1/2}(\mathrm{div}_{\Gamma_+},\Gamma_+)$ such that

$$T\boldsymbol{\omega} = \gamma_\tau^+(\boldsymbol{H}(\boldsymbol{\omega})).$$

Because of the fact that T maps Neumann data to Dirichlet data, it is called Neumann-to-Dirichlet map. We have to pick up such $\boldsymbol{\omega}$ which satisfies:

$$T\boldsymbol{\omega} = \vec{H} \tag{10}$$

in order to fulfill the second equation of (7) and thereby solve the problem. The operator eqn.(10) is uniquely solvable if (7) is uniquely solvable: One can understand (10) as the chain $\boldsymbol{\omega} \to \boldsymbol{H}(\boldsymbol{\omega}) \to \gamma_\tau^+(\boldsymbol{H}(\boldsymbol{\omega})) = \vec{H}$. The first mapping has the unique solution $\boldsymbol{H}(\boldsymbol{\omega}) \in \mathbf{H}(\mathbf{curl}, \Omega)$ due to Lax-Milgram theorem and the second mapping due to the **Theorem 1**.

We approximate $\mathbf{H}(\mathbf{curl}, \Omega)$ by a suitable finite element space $\boldsymbol{V}_h$ using Whitney elements. Let the finite dimensional space $\boldsymbol{V}_h^-$ approximate the trace space $\mathbf{H}_{\|,00}^{-1/2}(\mathrm{div}_{\Gamma_-}, \Gamma_-)$. The straightforward idea is to take basis functions $\boldsymbol{e}_i$ from this space and compute:

$$\begin{array}{rcll} \boldsymbol{H}_{\boldsymbol{e}_i} + \alpha\nabla\wedge\nabla\wedge\boldsymbol{H}_{\boldsymbol{e}_i} & = & \mathbf{0} & \text{in} \quad \Omega, \quad \text{for } \alpha \in \mathbb{R}_+ \\ \nabla\wedge\boldsymbol{H}_{\boldsymbol{e}_i}\wedge\boldsymbol{n} & = & \mathbf{0} & \text{on} \quad \Gamma_+ \qquad\qquad i = 1\ldots\dim(\boldsymbol{V}_h^-) \\ \nabla\wedge\boldsymbol{H}_{\boldsymbol{e}_i}\wedge\boldsymbol{n} & = & \boldsymbol{e}_i & \text{on} \quad \Gamma_- \end{array} \tag{11}$$

and to look for $\boldsymbol{\omega}$ in the form $\boldsymbol{\omega} \approx \sum_{\boldsymbol{e}_i\in\boldsymbol{V}_h^-} \alpha_i\boldsymbol{e}_i$, which yields $\boldsymbol{H}_1(\boldsymbol{\omega}) \approx \sum_{\boldsymbol{e}_i\in\boldsymbol{V}_h^-} \alpha_i\boldsymbol{H}_{\boldsymbol{e}_i}$. As $\boldsymbol{H}$ has to fulfill $\boldsymbol{H}\wedge\boldsymbol{n} = \vec{H}$ on Γ_+, we have the condition :

$$\sum_{\boldsymbol{e}_i\in\boldsymbol{V}_h^-} \alpha_i\,\boldsymbol{H}_{\boldsymbol{e}_i}\wedge\boldsymbol{n} \approx \vec{H} - \boldsymbol{H}_2\wedge\boldsymbol{n} \quad \text{on} \quad \Gamma_+. \tag{12}$$

This is a discrete form of (10). We solved (12) in two ways. The area of Γ_+ is usually much smaller than that of Γ_-. This is a consequence of the fact that we measure on a small part of the boundary. Thus, $\dim(\boldsymbol{V}_h^+) < \dim(\boldsymbol{V}_h^-)$ in case of a regular mesh, where $\boldsymbol{V}_h^+$ approximates $\mathbf{H}_{\|,00}^{-1/2}(\mathrm{div}_{\Gamma_+}, \Gamma_+)$. It results in an under-determined system (12). It can be easily cured by refining the mesh on Γ_+. So, we can suppose if necessary that $\dim(\boldsymbol{V}_h^+) \geq \dim(\boldsymbol{V}_h^-)$. One can pick some of the "answers" and gain a square matrix. We tried this for a thin box. More about this approach can be found in the section of numerical examples.

As $\dim(\boldsymbol{V}_h^+) > \dim(\boldsymbol{V}_h^-)$, the general approach yields the minimization of

$$F(\boldsymbol{\omega}) := \left\|\boldsymbol{H}(\boldsymbol{\omega})\wedge\boldsymbol{n} - \vec{H}\right\|^2_{\mathbf{H}_{\|,00}^{-1/2}(\mathrm{div}_{\Gamma_+},\Gamma_+)} + \text{regularization term}. \tag{13}$$

Hereafter we have to require from the solution $\boldsymbol{H}(\boldsymbol{\omega})$ and from the data $\vec{H}$ an additional regularity to be able to replace the norm $\|\,.\,\|_{\mathbf{H}_{\|,00}^{-1/2}(\mathrm{div}_{\Gamma_+},\Gamma_+)}$ by $\|\,.\,\|_{\mathbf{L}^2(\Gamma_+)}$. For example $\boldsymbol{H}(\boldsymbol{\omega}) \in \mathbf{H}^1(\Omega)$ and $\vec{H} \in \mathbf{L}^2(\Gamma_+)$. After the substitution we get :

$$F(\boldsymbol{\omega}) \approx \left\|\left(\sum_{i\in\boldsymbol{V}_h^-} \alpha_i\,\boldsymbol{H}_{\boldsymbol{e}_i} + \boldsymbol{H}_2\right)\wedge\boldsymbol{n} - \vec{H}\right\|^2_{\mathbf{L}^2(\Gamma_+)} + \text{regularization term}. \tag{14}$$

For the regularization term we can not use the classical choice of Tikhonov $\eta\,\|\boldsymbol{H}(\boldsymbol{\omega})\|^2_{\mathbf{L}^2(\Omega)}$, because $\boldsymbol{H}_{\boldsymbol{e}_i}$ are generally full vectors. This would result in a full regularization matrix of $\left(\boldsymbol{H}_{\boldsymbol{e}_i}, \boldsymbol{H}_{\boldsymbol{e}_j}\right)_{i,j}$ type. Let's take a regularization :

$$\eta \sum_{\boldsymbol{e}_i\in\boldsymbol{V}_h^-} \|\alpha_i\boldsymbol{H}_{\boldsymbol{e}_i}\|^2_{\mathbf{L}^2(\Omega)} \tag{15}$$

The minimum is reached in the solution of the following normal equations:

$$\sum_{\boldsymbol{e}_i\in\boldsymbol{V}_h^-} \alpha_i\,\boldsymbol{H}_{\boldsymbol{e}_i}\wedge\boldsymbol{n}\cdot\boldsymbol{H}_{\boldsymbol{e}_j}\wedge\boldsymbol{n} + \eta\alpha_j\left\|\boldsymbol{H}_{\boldsymbol{e}_j}\right\|_{\mathbf{L}^2(\Omega)} = (\vec{H} - \boldsymbol{H}_2\wedge\boldsymbol{n})\cdot\boldsymbol{H}_{\boldsymbol{e}_j}\wedge\boldsymbol{n}, \quad j = 0\ldots\dim(\boldsymbol{V}_h^-) \tag{16}$$

which are gained by differentiating (14) with respect to $\alpha_j, j = 0\ldots\dim(\boldsymbol{V}_h^-)$.

4. 2^{nd} APPROACH - ADJOINT METHOD

Let us consider the variational problem

Problem 2. *Find $\boldsymbol{\omega}$ on Γ_- which minimizes the following functional :*

$$F(\boldsymbol{\omega}) \quad := \quad \left\| \boldsymbol{H}(\boldsymbol{\omega}) \wedge \boldsymbol{n} - \vec{H} \right\|^2_{\mathbf{H}^{-1/2}_{\|,00}(\mathrm{div}_{\Gamma_+},\Gamma_+)} + \eta \, \|\boldsymbol{H}(\boldsymbol{\omega})\|^2_{\mathbf{H}(\mathbf{curl}\,,\Omega)}\,, \tag{17}$$

where the second term is a regularization item with a constant $\eta > 0$ and $\boldsymbol{H}(\boldsymbol{\omega})$ is the solution to the following problem :

$$\begin{array}{rcll} \boldsymbol{H}(\boldsymbol{\omega}) + \alpha \nabla \wedge \nabla \wedge \boldsymbol{H}(\boldsymbol{\omega}) & = & \boldsymbol{f} & in \quad \Omega,\ \alpha \in \mathbb{R}_+ \\ \nabla \wedge \boldsymbol{H}(\boldsymbol{\omega}) \wedge \boldsymbol{n} & = & \vec{C} & on \quad \Gamma_+ \\ \nabla \wedge \boldsymbol{H}(\boldsymbol{\omega}) \wedge \boldsymbol{n} & = & \boldsymbol{\omega} & on \quad \Gamma_-, \end{array} \tag{18}$$

where $\vec{H}$ and $\vec{C}$ are given boundary data and f is divergence free.

We suppose, that our problem is uniquely solvable (or at least solvable). As argued before, $\vec{H}$ as well as $\vec{C}$ have to belong to $\mathbf{H}^{-1/2}_{\|,00}(\mathrm{div}_{\Gamma_+},\Gamma_+)$. It is obvious that

$$\boldsymbol{H}(\boldsymbol{\omega}) = \boldsymbol{H}_1(\boldsymbol{\omega}) + \boldsymbol{H}_2. \tag{19}$$

The goal is to generate a minimizing sequence of solutions $\boldsymbol{H}(\boldsymbol{\omega})$ by the steepest descent method:

$$\boldsymbol{\omega}_{k+1} = \boldsymbol{\omega}_k - \alpha_k F'(\boldsymbol{\omega}_k) \tag{20}$$

starting with an initial guess $\boldsymbol{\omega}_0$ with a controlled step α_k. The derivative $F'(\boldsymbol{\omega})$ is understood in the weak sense, namely it is the first variation of $F(\boldsymbol{\omega})$, defined by

$$F(\boldsymbol{\omega} + \delta\boldsymbol{\omega}) - F(\boldsymbol{\omega}) = \langle F'(\boldsymbol{\omega}), \delta\boldsymbol{\omega} \rangle + \mathcal{O}(\|\delta\boldsymbol{\omega}\|^2_{\mathbf{H}^{-1/2}_{\|}(\mathrm{div}^0_-,-)}). \tag{21}$$

The implementation of the steepest descent method requires an explicit form of $F'(\boldsymbol{\omega})$, which will be derived in the next subsection.

4.1 Derivation of $F'(\boldsymbol{\omega})$

It can be shown that the minimum of the quotient norm

$$\|\boldsymbol{m}\|^2_{\mathbf{H}^{-1/2}_{\|,00}(\mathrm{div}_{\Gamma_+},\Gamma_+)} := \inf_{\substack{\boldsymbol{u} \in \mathbf{H}(\mathbf{curl}\,,\Omega) \\ \boldsymbol{u} \wedge \boldsymbol{n}|_{\Gamma_+} = \boldsymbol{m}}} \|u\|^2_{\mathbf{H}(\mathbf{curl}\,,\Omega)}$$

is reached in the solution to the following problem:

$$\begin{array}{rcll} \boldsymbol{v_0} + \nabla \wedge \nabla \wedge \boldsymbol{v_0} & = & \mathbf{0} & \text{in} \quad \Omega \\ \nabla \wedge \boldsymbol{v_0} \wedge \boldsymbol{n} & = & \mathbf{0} & \text{on} \quad \Gamma_- \\ \boldsymbol{v_0} \wedge \boldsymbol{n} & = & \boldsymbol{m} & \text{on} \quad \Gamma_+. \end{array} \tag{22}$$

We have $\nabla \wedge \boldsymbol{v}_0 \in \mathbf{H}(\mathbf{curl}\,,\Omega)$, moreover $\nabla \wedge \boldsymbol{v}_0 \in \mathbf{H}_{0,\Gamma_-}(\mathbf{curl}\,,\Omega)$. Particularly, **Theorem 1** says that $\nabla \wedge \boldsymbol{v}_0$ has the trace $\gamma^{+,0}_\tau(\nabla \wedge \boldsymbol{v}_0) \in \mathbf{H}^{-1/2}_{\|}(\mathrm{div}^0_{\Gamma_+},\Gamma_+)$.

If we denote by $v_0(\boldsymbol{m})$ the solution of (22), we can rewrite the functional (17) as:

$$F(\boldsymbol{\omega}) \quad := \quad \left\| v_0(\boldsymbol{H}(\boldsymbol{\omega}) \wedge \boldsymbol{n} - \vec{H}) \right\|^2_{\mathbf{H}(\mathbf{curl}\,,\Omega)} + \eta \, \|\boldsymbol{H}(\boldsymbol{\omega})\|^2_{\mathbf{H}(\mathbf{curl}\,,\Omega)}\,. \tag{23}$$

We recall that $\boldsymbol{v}_0(\boldsymbol{m})$ is linear in $\boldsymbol{m}$ and $\boldsymbol{H}_1(\boldsymbol{\omega})$ is linear in $\boldsymbol{\omega}$!

We proved in a forthcoming paper, that $(\boldsymbol{v}_0(\boldsymbol{H}(\boldsymbol{\omega}) \wedge \boldsymbol{n} - \vec{H}) =: \boldsymbol{v}_0(\boldsymbol{\omega}, \vec{H}))$

$$\begin{array}{rcl} F(\boldsymbol{\omega} + \delta\boldsymbol{\omega}) - F(\boldsymbol{\omega}) & = & {}_{\gamma,\Gamma_-}\left\langle \delta\boldsymbol{\omega}, 2\eta\, \pi^-_\tau(\boldsymbol{H}_1(\boldsymbol{\omega})) + 2\pi^-_\tau(\boldsymbol{v}_0(\boldsymbol{\omega}, \vec{H})) \right\rangle_{\pi,00,\Gamma_-} \\ & & + \mathcal{O}(\|\delta\boldsymbol{\omega}\|^2_{\mathbf{H}^{-1/2}_{\|}(\mathrm{div}^0_-,-)}), \end{array}$$

where ${}_{\gamma,\Gamma_-}\langle\,\cdot\,,\,\cdot\,\rangle_{\pi,00,\Gamma_-}$ denotes the duality between $\mathbf{H}^{-1/2}_{\|}(\mathrm{div}^0_{\Gamma_-},\Gamma_-)$ and $\mathbf{H}^{-1/2}_{\perp,00}(\mathrm{curl}_{\Gamma_-},\Gamma_-)$ [4]. Note, that **Theorem 1** says that $\nabla \wedge \boldsymbol{H}_1 \in \mathbf{H}_{0,\Gamma_+}(\mathbf{curl}\,,\Omega)$ has the trace $\gamma^{-,0}_\tau(\nabla \wedge \boldsymbol{H}_1) \in \mathbf{H}^{-1/2}_{\|}(\mathrm{div}^0_{\Gamma_-},\Gamma_-)$. Thus $\delta\boldsymbol{\omega} = \gamma^{-,0}_\tau(\nabla \wedge \boldsymbol{H}_1(\delta\boldsymbol{\omega})) \in \mathbf{H}^{-1/2}_{\|}(\mathrm{div}^0_{\Gamma_-},\Gamma_-)$.

By comparison with definition (21) we get :

$$F'(\boldsymbol{\omega}) = 2\eta\,\pi_\tau^-(\boldsymbol{H}_1(\boldsymbol{\omega})) + 2\pi_\tau^-(\boldsymbol{v}_0(\boldsymbol{\omega},\vec{H})). \tag{24}$$

We find $\boldsymbol{v}_0$ in a weak sense, as $\boldsymbol{v}_0 = \boldsymbol{v}_{\Gamma_1} - \boldsymbol{v}_{\Gamma_2} + \boldsymbol{v}$, where $\boldsymbol{v}_{\Gamma_1} \in \mathbf{H}(\mathbf{curl}\,,\Omega)$ is any vector field having the trace $\boldsymbol{H}(\delta\boldsymbol{\omega})\wedge\boldsymbol{n}$ on Γ_+ (such $\boldsymbol{v}_{\Gamma_1}$ exists, since γ_τ is surjective, namely $\boldsymbol{H}(\delta\boldsymbol{\omega})$ has this trace). Further, $\boldsymbol{v}_{\Gamma_2} \in \mathbf{H}(\mathbf{curl}\,,\Omega)$ is any vector field having the trace $\vec{H}$ on Γ_+. Since $\boldsymbol{v}_0$ is a solution to (22), we can write :

$$(\nabla\wedge\boldsymbol{v_0},\nabla\wedge\boldsymbol{\phi}) + (\boldsymbol{v_0},\boldsymbol{\phi}) = \mathbf{0} \quad \forall\boldsymbol{\phi}\in\mathbf{H}_{0,\Gamma_+}(\mathbf{curl}\,,\Omega).$$

Thus, if $\boldsymbol{v}$ is the solution to the following problem:

Problem 3. *Find* $\boldsymbol{v}\in\mathbf{H}_{0,\Gamma_+}(\mathbf{curl}\,,\Omega)$ *such that*

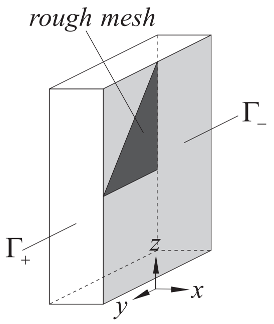
$$\begin{aligned}(\nabla\wedge\boldsymbol{v},\nabla\wedge\boldsymbol{\phi}) + (\boldsymbol{v},\boldsymbol{\phi}) = (-\nabla\wedge\boldsymbol{H}(\delta\boldsymbol{\omega}) + \nabla\wedge\boldsymbol{v}_{\Gamma_2},\nabla\wedge\boldsymbol{\phi}) + (-\boldsymbol{H}(\delta\boldsymbol{\omega})+\boldsymbol{v}_{\Gamma_2},\boldsymbol{\phi})\\ \forall\boldsymbol{\phi}\in\mathbf{H}_{0,\Gamma_+}(\mathbf{curl}\,,\Omega),\end{aligned} \tag{25}$$

then $\boldsymbol{v}_0 = \boldsymbol{H}(\delta\boldsymbol{\omega}) - \boldsymbol{v}_{\Gamma_2} + \boldsymbol{v}$ is a solution to (22).

The theoretical bases with the proofs will be published in a forthcoming paper. The optimal choice of η will also be studied. One can prove that (20) with control of a step size defines a relaxation sequence converging to absolute minimum point of $F(\omega)$. The idea of the proof is based on the fact that $F(\omega)$ is strictly convex and coercive in suitable function spaces [11].

5. NUMERICAL EXAMPLES

In this section we present some computations to demonstrate the MOA and the AM, which have been described above. Let $\Omega = (0,h)\times(0,1)\times(0,1), h\in(0,1]$. In our investigations we will show the dependence of the MOA on the thickness h of the domain. In the experiments we put $\alpha = 1$. For the space discretization we use the edge finite element method [2]. The BICGSTAB solver is used to solve all resulting linear systems.



In our examples we take

$$\boldsymbol{H} = (\sin(z),\sin(x),\sin(y)) \tag{26}$$

as exact solution.

First, we consider the MOA. The simple geometry of a box allows us to arrive directly at a square linear system which has to be solved. We will work on two uniform meshes. A rough mesh is used to describe the system (12). The answers $\boldsymbol{H}_{\boldsymbol{e}_i}$ are computed on a fine mesh. The impulses are functions:

$$\begin{aligned}\boldsymbol{e}_{1,i} &= \begin{cases}(0,1,0) & \text{on } i\text{-th } \triangle \text{ of } \Gamma_-\\ 0 & \text{otherwise,}\end{cases}\\ \boldsymbol{e}_{2,i} &= \begin{cases}(0,0,1) & \text{on } i\text{-th } \triangle \text{ of } \Gamma_-\\ 0 & \text{otherwise,}\end{cases}\end{aligned} \tag{27}$$

whose supports are faces of tetrahedra of the rough mesh on Γ_-. Let's recall that

$$\boldsymbol{V}_h^- = \{\boldsymbol{e}_{1,i},\boldsymbol{e}_{2,i}\}.$$

Thus, the number of impulses is twice the number of triangles on Γ_-. We will collect the answers only on the opposite face of the box in order to generate a square linear system of algebraic equations. A successive multiplication of (12) by test functions $\boldsymbol{f}_j$ yields

$$\sum_{i\in\boldsymbol{V}_h^-}\alpha_i\,(\boldsymbol{H}_{\boldsymbol{e}_i}\wedge\boldsymbol{n})\cdot\boldsymbol{f}_j \approx (\vec{H}-\boldsymbol{H}_2\wedge\boldsymbol{n})\cdot\boldsymbol{f}_j \qquad \text{on} \quad \Gamma_+, \tag{28}$$

where $\boldsymbol{f}_j$ are the same functions as (27) but they are living on triangles of opposite face of the box. Due to the fact that our mesh is regular, we obtain exactly the same number of answers as impulses. Thus we gained a square matrix

$$\mathbf{M} = (\boldsymbol{H}_{\boldsymbol{e}_i}\wedge\boldsymbol{n}\cdot\boldsymbol{f}_j) \quad i,j = 1\ldots\dim(\boldsymbol{V}_h^-). \tag{29}$$

The resulting matrix is full in the sense that almost all the values are nonzero. There arises a natural question: “How sensitive is the proposed method to the thickness h of the domain”? One can expect that the numerical error associated with a direct problem will increase with increasing h. A similar scheme but for a corrosion detection has been studied in [7–9]. The authors showed also theoretically that a thick domain causes an instability of proposed numerical approach.

We observed this instability in the next numerical example. The width of domain h went from 0.005 to 0.4. We computed on a regular triangulation $1 \times 2 \times 2$ twice uniformly refined. The resulting mesh had 2583 DOFs. The results are depicted in Figure 2. The e in all figures and tables corresponds to the relative L^2-error between the numerical solution and the exact one. The numerical solution is very

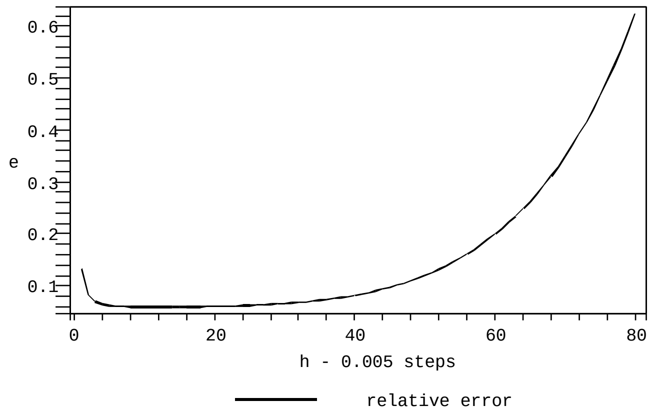


Figure 2: MOA: Sensitivity to the thickness of a domain.

sensitive to the thickness of domain as it was expected. As h decreases, the accuracy increases. The drop of the accuracy when the domain is very thin is due to the irregularity of the mesh, which overrides positive effects of the narrowing.

Further, the sensitivity to noise is studied. Here $h = 0.1$. All other parameters are the same as in the former computation. The tested range of noise-level is from 0 to 2 %. The results are presented in Figure 3. The sensitivity to perturbation of data demonstrates almost linear trend. The 2 % noise-level in data causes approximately 2% raise in the relative error.

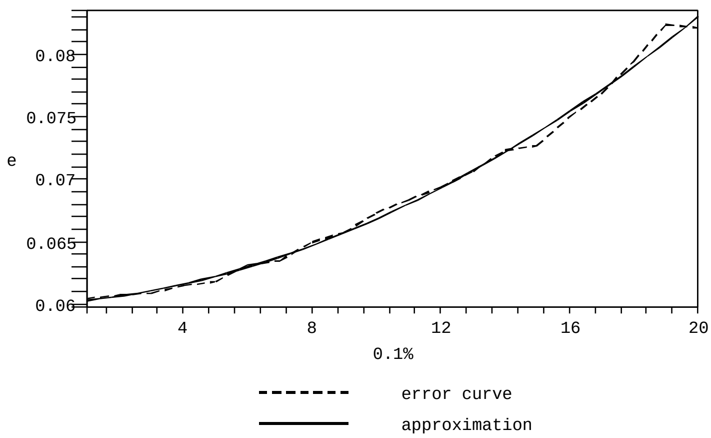


Figure 3: MOA: Sensitivity to perturbation.

Next, we studied the convergence of the method. Here $h = 0.1$ and no noise is present. The results are collected in Table 1. The n is the rank of matrix $\mathbf{M}$ and it represents the number of DOFs. The symbol D in the table stands for a divergence, i.e., the relative error of a solution was not reasonable in such cases. The results support the idea of computation on rough and fine meshes. In the case of a computation of answers $\boldsymbol{H}_{\boldsymbol{e}_i}$ and a linear system (28) on the same mesh, a very ill-conditioned matrix $\mathbf{M}$ is generated (for example $n = 16$ and DOFs $= 57$ from Table 1).

Next, we mention an application of (16). We considered also this scheme. It can be used to acquire

$n\backslash$ DOFs	57	387	2583	18495	139407
16	D	0.111271	0.060407	0.032195	0.018508

Table 1: MOA relative error e, $h = 0.1$, n - rank of matrix $\mathbf{M}$.

solution on a thick domain, too. In spite of the fact that it looks promising, some a posteriori techniques for updating η have to be applied. This will be subject of our future investigation.

Finally, we present some numerical results for the AM. All evaluations are done on the unit cube. First, the sensitivity to noise-level of data is tested. The initial guess $\boldsymbol{\omega}_0 = 0$ is taken. The results for the regularization parameter η equal to 0.0, 0.01, 0.1 and 1 are depicted in Figure 4. These are

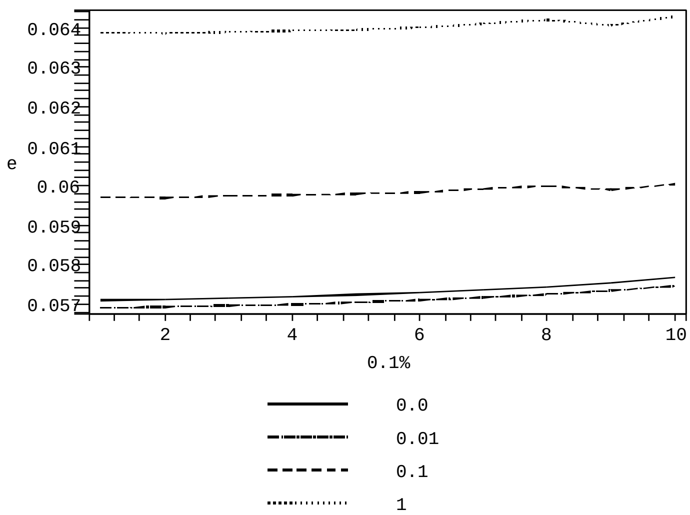


Figure 4: AM sensitivity to noise level for different η.

slightly surprising. The $\eta = 0.01$ appears to be the best choice, but $\eta = 0.0$ behaves very well, too. In fact, the difference between them seems to be negligible. The results for $\eta = 0.1$ and $\eta = 1$ lag behind. This can be misleading and one can think that the regularization is not needed. But, from the theoretical point of view, the regularization term $\eta \left\| \boldsymbol{H}(\boldsymbol{\omega}) \right\|^2_{\mathbf{H}(\mathbf{curl},\Omega)}$ was crucial to prove the strict convexity of the cost functional in suitable function spaces. Moreover, the regularity of exact solution $\boldsymbol{H} = (\sin(z), \sin(x), \sin(y))$ also plays a part. Worse results should be expected for real data.

In Table 2 the reader can find a numerical justification, that the proposed method acts feasible. The

DOFs	19	117	721	4905	35929
e	0.333953	0.181730	0.106885	0.056909	0.029173
iter	52	52	45	17	16
time/sec	1	3	15	101	1115

Table 2: AM, $\eta = 0.01$.

CPU-times of computations are just for information. These can depend strongly on the implementation of the method. More important are the numbers of iterations, more precisely, the number of direct problems needed to approach the solution. The algorithm terminates if the difference of the last values of the functional is less than 10^{-6}.

In all tests above, Γ_- was just one face of the box. It does not correspond to reality. As mentioned before, usually $\dim(\boldsymbol{V}_h^+) < \dim(\boldsymbol{V}_h^-)$ in case of a regular mesh. As we work on regular meshes, we could not test MOA for these more realistic situations. But the AM is quite independent on the mesh structure. Let Γ_- consists of 5 faces and we have measurements only on one face of the unit cube. The AM behaves good also in this situation. We applied it on a regular mesh with 4905 DOFs without any noise. The reached accuracy is almost identical with that in Table 2, $e = 0.058293$, but the method needed two times more steps to approach solutions - 34 steps.

6. CONCLUSIONS

We presented two possible methods to solve a boundary identification problem for the eddy-current model in the case $\mu_r >> 1$ outside of domain. The first one, the MOA, was designed for a thin box, but can be extended to any domain by constructing the system of normal equations and adding a regularization term to it. However, to construct this system we have to impose more regularity on the data and the solution. The main idea was to compute on two meshes, a rough and a fine one. It possesses quite strong numerical stability to noise, but is not easy to apply.

The second method, the AM or the descent steepest method is much easier to apply. There is no need to impose any non-natural regularity neither on the data nor on the solution. All computations were done on the same mesh. In spite of that fact it possesses a stronger numerical stability to noise than the MOA. The method is an ideal candidate for an application of a quasi-Newton method to accelerate the convergence.

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